INSTITUT FÜR THEORETISCHE PHYSIK UND ASTROPHYSIK
Christian-Albrechts-Universität zu Kiel[1]

Finding the Standard Model of Particle Physics A Combinatorial Problem

Jan-Hendrik Jureit [2, 3, 5] Christoph A. Stephan [2, 4, 5]

Abstract

We present a combinatorial problem which consists in finding irreducible Krajewski diagrams from finite geometries. This problem boils down to placing arrows into a quadratic array with some additional constrains. The Krajewski diagrams play a central role in the description of finite noncommutative geometries. They allow to localise the standard model of particle physics within the set of all Yang-Mills-Higgs models.

March 2005

[1] Leibnizstraße 15, D-24098 Kiel, Germany
[2] Université de Provence
[3] jureit@cpt.univ-mrs.fr
[4] stephan@cpt.univ-mrs.fr
[5] Centre de Physique Théorique Unité Mixed de Recherche (UMR) 6207 du CNRS et des Universités Aix-Marseille 1 et 2 Sud Toulon-Var, Laboratoire affilié à la FRUMAM (FR 2291)

1 Introduction

In this paper we present an algebraic problem which has its roots in noncommutative geometry. Given the set of all square matrices $M \in M_n(\mathbb{Z})$, with integer entries and for fixed n. These matrices are called multiplicity matrices. Define a partial order in $M_n(\mathbb{Z})$ by $M \geq M'$ when M_{ij} and M'_{ij} have the same sign and $|M_{ij}| \geq |M'_{ij}|$ for all $i, j = 1, \ldots, n$. The task is now to find all the minimal multiplicity matrices which obey two further conditions. First, the determinant of M has to be non-zero. The second condition asserts that for any non vanishing matrix element there has to exist a second element of oposite sign in the same row or in the same column. It is straight forward to translate this problem into a combinatorial problem using diagramtic language with arrows as basic elements. We will give a simple set of rules how to fit arrows into a quadratic diagram body. The resulting diagrams are called Krajewski diagrams. They can be reduced by combining or "clipping" the arrows in the diagram in accordance with the rules. To each Krajewski diagram a multiplicity matrix will be associated. Our aim will be to find those Krajewski diagrams with a minimal number of arrows that are also as much reduced as possible while the determinant of the multiplicity matrix is non-zero. These diagrams will be called irreducible.

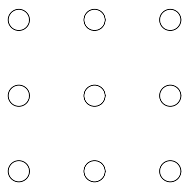
The rules to construct Krajewski diagrams follow directly from Alain Connes noncommutative geometry [3]. They are due to a special class of noncommutative geometries called almost-commutative geometries. These unify the general theory of relativity with the classical field theory of the standard model of particle physics [2]. It is possible to give a complete classification of almost-commutative geometries [5] which can be narrowed under addition of some physical assumptions to a classification of a class of Yang-Mills-Higgs theories. Within these theories the standard model of particle physics takes a most prominent place [1]. For a thorough introduction into the physical application of almost-commutative geometry we refer to [4].

2 Constructing the Diagrams

Our task will now be to construct an algorithm that finds the irreducible Krajewski diagrams. To begin with we will give the basic rules that follow from the axioms of spectral triples and from the requirement of irreducibility. These rules will allow us to generate the diagrams and to perform all the necessary operations.

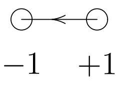
2.1 Basic Rules

The diagram body: The diagram body is the basis to construct diagrams from. It is a quadratic array of a given size with circles indicating the intersection points or the rows and the columns. These circles have no meaning on their own but are there to guide the eye of the reader. The size of the diagram body will be fixed in advance.



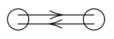
3×3 diagram body

The basic arrow: The basic simple arrow connects two circles of the diagram body. With the end points of an arrow we associate two numbers which are called the chirality. This term stems from noncommutative geometry and corresponds to the chirality of particle physics. A simple arrow points from chirality $+1$ to chirality -1.



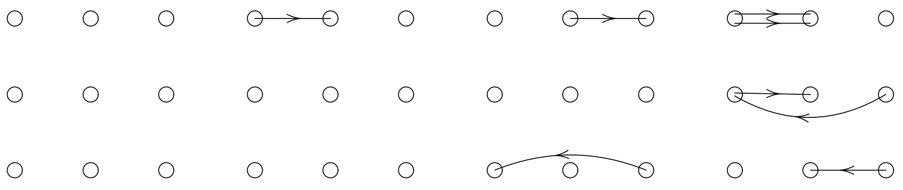
Simple Arrow

Constructing a basic diagram: To construct a basic diagram we put n horizontal arrows in an arbitrary way into the body. Only an antiparallel arrangement of arrows is forbidden.



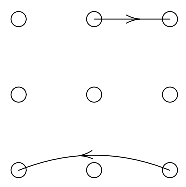
Forbidden antiparallel arrows

Diagrams are sorted by their total number of arrows. The empty diagram (i.e. the diagram body) is said to lie in level 0, the diagrams with one arrow are said to lie in level 1 and so on. For 3×3 diagrams some examples of possible diagrams are:



Level 0 Level 1 Level 2 Level 5

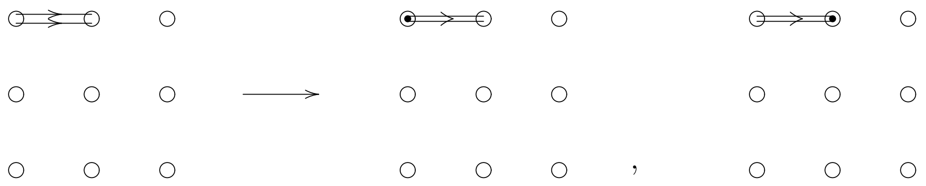
The Multiplicity Matrix: To every diagram a matrix, called the *multiplicity matrix M*, is associated. It is a direct translation of the diagram by taking the values of the chiralities at each end of an arrow. These values are then written into a square matrix of the same size as the diagram and at the same position as they appear in the diagram. The multiplicity matrix is then this matrix plus its transposed as shown in the following example:



$$\longrightarrow \quad M = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 0 \\ -2 & 0 & -2 \end{pmatrix}$$

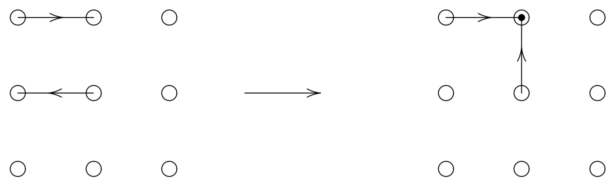
The multiplicity matrix of a diagram

Clipping two arrows: There are two ways to reduce a diagram. The first is by combining or *clipping* two arrows at a common point with common chirality or by building a corner again with arrow heads of common chirality. This creates a multiple arrow. Building a corner is achieved by transposing one of the arrows (reflecting it on the main diagonal of the diagram) which are to be clipped.



$$M = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow M = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Building a double arrow by clipping two parallel arrows

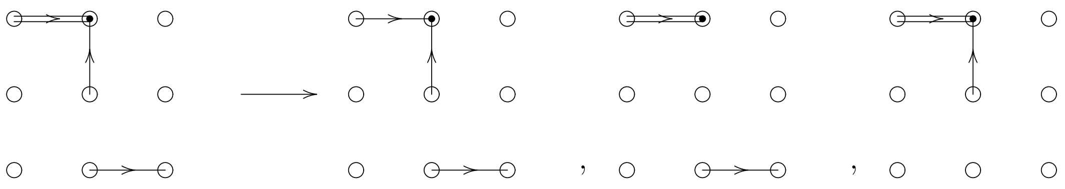


$$M = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Building a corner by clipping an arrow and a transposed arrow

The procedure to clip arrows extends in a straight forward way to more than two arrows. One only has to keep in mind to clip the arrows always at the same point in each connected component. Even two multiple arrows can be combined at their clipping points. Clipping arrows will not change the level a diagram belongs to.

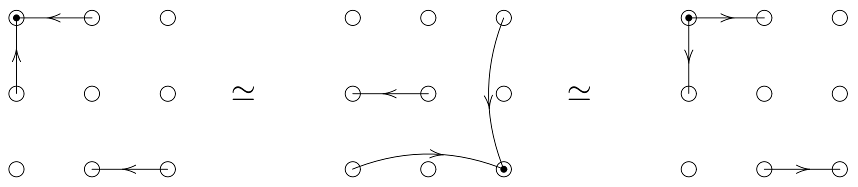
Erasing an arrow from a diagram: The second way to reduce a diagram is by erasing an arrow. Erasing an arrow from a diagram is completely trivial and lowers the level of the diagram by one. A simple arrow will be deleted and an arrow which is clipped to another (multiple) arrow will be deleted while leaving the element of the multiplicity matrix at clipping point unaltered:


All possibilities to erase one arrow from a diagram

Irreducible Diagrams: A diagram is said to be irreducible if it satisfies two conditions. First, the determinant of the multiplicity matrix has to be non-zero. Second, reducing the diagram by clipping or erasing arrows in all possible ways until one is left with the bare diagram body will on the way always produce diagrams with $\det M = 0$.

The Labels: Every diagram carries a label indicating the reducibility. An ‘i’ stands for an irreducible diagram (the determinant of the multiplicity matrix M is necessarily non-zero). Diagrams which carry an ‘r’ can be reduced to a diagram equipped with an ‘i’ or an ‘r’ ($\det M$ is of no importance). The label ‘0’ is carried by diagrams where the determinant of the multiplicity matrix is zero and which cannot be reduced to diagrams equipped with an ‘r’ or an ‘i’. The labels are of internal use only and we usually do not write them explicitely.

Equivalence of Diagrams: The physical theory following from a diagram is not dependent on the order of the rows and columns or on a reversal of all arrows at once. Permuting the rows and columns together or reversing the arrows does not change the absolute value of the determinant of the multiplicity matrix. Consequently diagrams that differ only by permutations of columns and rows or by reversing the arrows will be regarded as equivalent. All these diagrams build an equivalence class and only one representative will be used for further operations. As an example the following three diagrams are equivalent:


Three equivalent diagrams

With regard to several thousands of diagrams it becomes a very time consuming procedure to directly test all diagrams for possible equivalence. To compare two diagrams the set of arrows of one of the diagrams has to be permuted in all possible ways and reversed. All the resulting diagrams have to be compared to the other diagram. Sorting the diagrams of one level into equivalence classes by direct comparison is most uneconomical.

But there exist invariants of the diagrams. These will provide us with a quicker way to decide whether two diagrams are inequivalent or not. Only if all these invariants prove to be equal for two diagrams it becomes necessary to decide the question of equivalence by permuting and reversing the arrows:

1. Compare directly the arrows of the two diagrams. If the arrows coincide the diagrams are equal and thus equivalent.

2. Compare the absolute values of the determinants of the multiplicity matrices. If they do not coincide the diagrams are inequivalent.

3. Compare the absolute values of the traces of the multiplicity matrices. If they do not coincide the diagrams are inequivalent.

4. Compare the Chern character of the multiplicity matices. If they do not coincide the diagrams are inequivalent. This Chern character of the multiplicity matrix is defined as $\det(1 + M^2)$.

5. If the two diagrams are not inequivalent by virtue of points 1 to 4, permute and reverse the set of arrows and compare the resulting diagrams.

It is remarkable that the operations of point 1 to 4 are powerful enough to find out in most cases whether two diagrams are inequivalent. This reduces the computation time most considerably.

The Game: With the basic rules formulated above our combinatorial game is to find all the equivalence classes of irreducible Krajewski diagrams for a given size of the diagram body and for a given maximal number of arrows. This seemingly simple task will produce complex structures which we will call *diagram nets*. These nets fall into two categories, the main-nets and the clip-nets. We will now present an algorithm to generate these nets and to find the embedded irreducible Krajewski diagrams.

3 The Algorithm

The algorithm to find the irreducible Krajewski diagrams can be divided into four subalgorithms. Every subalgorithm stands independently but builds up on the data produced by its predecessor. Summarised quickly the subalgorithms achieve the following: The first subalgorithm, dubbed the "main-net subalgorithm" creates a diagram net consisting of all equivalence classes of diagrams with up to N arrows. The "clip-net subalgorithm" creates a diagram net of equivalence classes from every element in the main-net, reducing each diagram by clipping the contained arrows in all possible ways. The third subalgorithm which we call the "label subalgorithm" checks whether the elements of a clip-net represent an irreducible diagram and sets the labels 'i', 'r' and '0' accordingly. Since the third subalgorithm can only see the elements in a single clip-net the equivalence classes carrying an 'i' or a '0' might still be reducible to an element one level below by erasing an arrow. So the last subalgorithm, the "label correction subalgorithm" checks these diagrams on their reducibility and changes their label to 'r' if necessary. The output of the whole algorithm are the diagrams labeled with an 'i' representing irreducible Krajewski diagrams.

3.1 The Main-Net Subalgorithm

The "main-net subalgorithm" fills the diagram body with up to N simple arrows. The maximal number of arrows, i.e. the maximal level, is chosen before. The arrows are put into the diagram body horizontally in every possible way, excluding only antiparallel arrows. Starting with the empty diagram body level 1 is filled with one horizontal arrow. The resulting diagrams are checked if they are equivalent and only one representative from each equivalence class is kept. In the same way every following level is built from the representative diagrams of the equivalence classes by adding one arrow. Each equivalence class is connected to its predecessors and its successors. Usually an equivalence class has several predecessors.

The subalgorithm to generate the main-net is depicted in flow chart 1 in figure 3. Some explanations to read the flowchart are necessary:

1. N is the maximal number of arrows and is put in by hand.

2. L is the level index from which the diagrams are taken.

3. D_L is an element of the ordered set that contains one representative of each equivalence class on level L. This kind of set is a basic notion in our flow charts. The specific order in the set is arbitrary, its only raison d'être is that the set can be run through from the first to the last element. This notion of ordered set has as its aim to unclutter the flow charts and to eliminate superfluous indices.

4. f_{D_L} is an element of the ordered set that contains all the operators which put one simple arrow into the diagram D_L in an allowed way. Each of these ordered sets is to be understood to belong to the specific diagram D_L that is processed in the current loop.

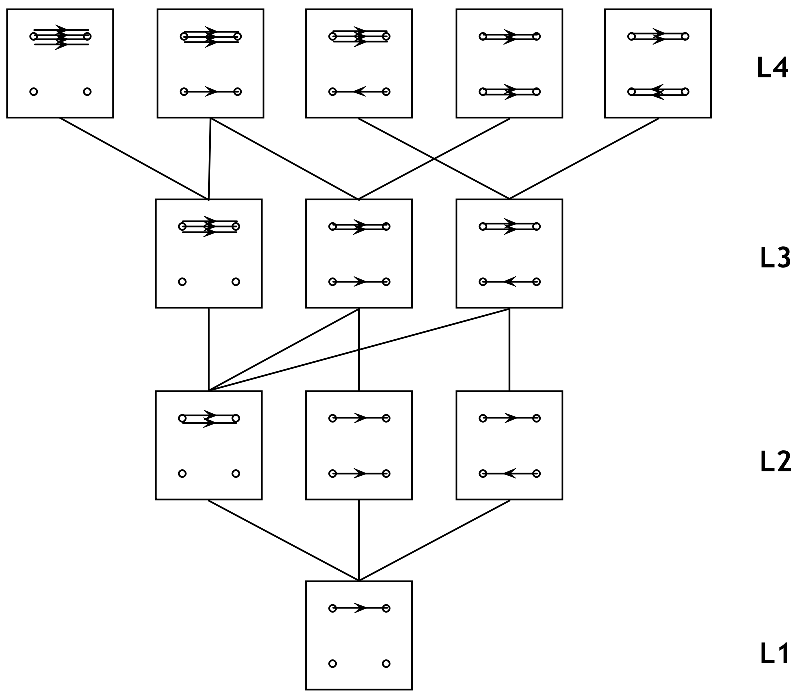


Figure 1: A main-net for a 2×2 diagram with up to four arrows

5. D_{L+1} is the diagram that is generated by putting in an arrow by virtue of the operator f_{D_L}.

6. The equivalence of diagrams is checked making use of the definitions and invariants specified in section 2.1.

An example of a main-net for a 2×2 diagram body with up to four arrows is given in figure 1.

3.2 The Clip-Net Subalgorithm

This algorithm creates a clip-net from each main-net diagram by clipping the arrows in all possible ways. The main-net diagram is said to lie in clip level 0. The diagrams where two arrows have been clipped are said to be in clip level 1, and so on. Each of the diagrams in the clip-nets is again gathered in equivalence classes which are connected with regard to their predecessors and successors. Here a technicality from the noncommutative geometry comes in: Whenever a corner is built by clipping the diagram has to 'forget' its predecessor. For a mathematical justification of this rule we refer to [1]. The maximal number of clip levels is determined by the number of arrows in the main-net diagram. If there are L arrows in a diagram there are up to $L - 1$ ways to clip these arrows.

The subalgorithm to generate the clip-net is depicted in flow chart 2 in figure 4. The explanations to read the flowchart are:

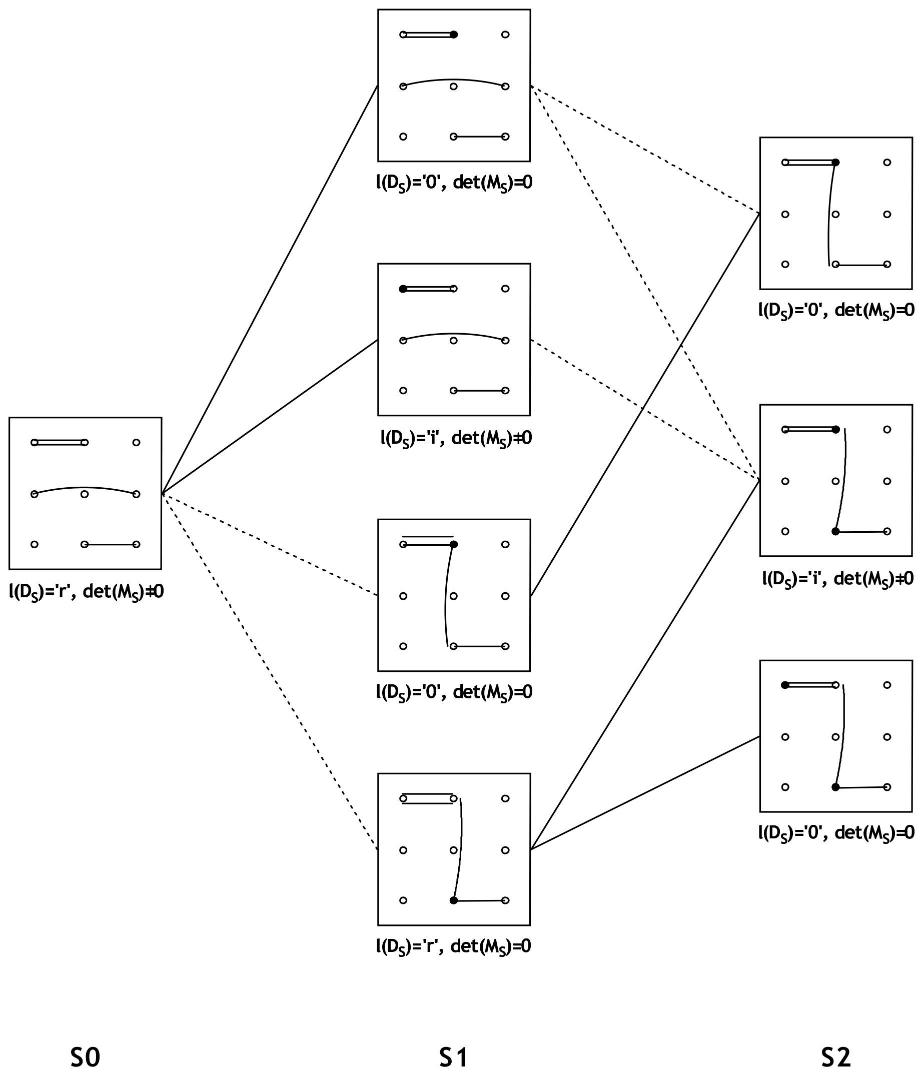


Figure 2: A simplified clip-net with 4 arrows and an example for the labelling

1. The representative of the equivalence class from the main-net is a member of the clip-net. It is the only element on clip level 0.

2. S is the clip level index. It can go up to $L-1$.

3. D_S is an element of the ordered set of representatives of the equivalence classes of the diagrams in clip-net S.

4. C_{D_S} is an element of the ordered set of operators which clip pairs of arrows in the diagram D_S, if possible. This set can be empty if it is impossible to clip any arrows.

5. D_{S+1} is the diagram produced by applying C_{D_S} to D_S, i.e. by clipping two arrows in D_S.

A simple example of such a clip-net is given by figure 2. Here we have left out the arrows to visualise only the basic structure of a clip-net. The dashed lines represent the forgotten connections due to the building of a corner.

3.3 The Label Subalgorithm

This subalgorithm assigns to each equivalence class in a clip-net one of the labels ‘i’, ‘r’ or ‘0’ (one should not forget that the elements of the main-net are included in the clip-net). For the labels ‘i’ and ‘0’ this assignment is provisional since it might be changed by the “label correction subalgorithm”. The subalgorithm starts with the highest clip level since here the most reduced diagrams are situated. Here the labels are set to ‘i’ or ‘0’ depending on the determinant of the multiplicity matrix of the diagram being non-zero or not. Then the algorithm descends one by one to clip level 0 setting the labels according to the determinant to ‘i’ and ‘0’ or setting the label to ‘r’ if the diagram is connected to a successor with label ‘i’ or ‘r’.

The subalgorithm to label the clip-net is depicted in flow chart 3 in figure 5. The explanations to read the flowchart are:

1. If S is the highest clip level the command “Get label of D_{S+1}” will always give label ‘0’ the since clip level $S+1$ does not exist and consequently there are no diagrams D_{S+1}.

2. $l(D_S)$ is the label of the element D_S and takes values ‘i’, ‘0’ or ‘r’.

3. $\mathrm{Det}(M_S)$ represents the determinant of the multiplicity matrix of D_S.

As an example for the labelling of a clip-net we also refer to figure 2. The labels and the determinants are only examples.

3.4 The Label Correction Subalgorithm

Since it may be possible to reduce a diagram carrying label ‘i’ or ‘0’ to a diagram with label ‘i’ or ‘r’ by erasing an arrow, it is necessary to correct these labels. It is evident that only the clip-nets of the predecessors of the main-net diagram are of importance for relabelling. The subalgorithm to relabel the clip-net is depicted in flow chart 4 in figure 6. The explanations to read the flowchart are:

1. $D_S^{0,i}$ is an element of the ordered set of *all* diagrams in the clip-net with label ‘i’ or ‘0’. The relations among the elements in the clip-net are of no importance since the diagrams are directly reduced to diagrams in clip-nets one level below.

2. $R_{D_S^{0,i}}$ is an element of the ordered set of operators that erase one arrow from the diagram $D_S^{0,i}$.

3. $l(D_S^{0,i})$ is the label of the element $D_S^{0,i}$ and takes values ‘i’, ‘0’ or ‘r’.

3.5 Assembling the Algorithm

The main algorithm is composed of the four subalgorithms. Its input is the size of the diagram body and the maximal number of arrows N. The assembly of the algorithm is now straight foreward:

Step 1: Use the main-net subalgorithm to create the main-net up to level N.

Step 2: Create for each element in the main-net its clip-net with the clip-net subalgorithm.

Step 3: Starting with level 1 label all clip-nets on this level with the label subalgorithm and afterwards check the labels with the label correction algorithm. Repeat this procedure successively for each level up to N.

Step 4: Print out all the elements with label 'i'. These are the equivalence classes representing irreducible Krajewski diagrams.

This algorithm runs very quickly on a recent PC with computation times of ca. 20 minutes for 4×4 diagrams and five levels. But the complexity of the nets with their clip-nets grows rapidly with the size of the diagram bodies and the maximal number of arrows.

4 Open Problems

There are two major open problems. The first problem is the question how to choose the maximal number N of arrows to be put into a diagram body. We know that the total number of irreducible Krajewski diagrams is finite, see [1], but we do not know the exact number of such diagrams. Up to now we adopted the simple conjecture that if no diagrams with label 'i' appear in two successive levels (say level 4 and level 5 for four algebras), we considered the maximal number of irreducibles as reached. But there is still a possibility of "islands" of irreducible Krajewski diagrams in higher levels. Although we have checked all the elements of the diagram net for 3×3 diagram bodies and can be sure that no more than three levels are necessary, already for 4×4 diagram bodies this is no longer possible. We are therefore looking for a criterion to determine the maximal level for a given size of the diagrams.

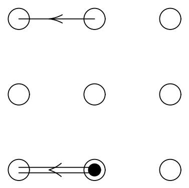
The second problem is the rapid increase of computational time and of memory required by the algorithm. We quickly exceed the capabilities of an ordinary personal computer and it will probably be necessary to use some sort of parallel computer or cluster to go beyond 5×5 diagram bodies.

5 Statement of the Result and Conclusions

We presented an algorithm to find irreducible Krajewski diagrams. The rules given here quickly produce very complex nets of diagrams in which the irreducibles are embedded.

One of the most surprising features of these nets is that for 3×3 and 4×4 diagram bodies only some twenty irreducible Krajewski diagrams appear. A complete list of these diagrams can be found in [1]. But already 5×5 diagram bodies produce several hundreds of irreducible diagrams. The order of magnitude of the total number of irreducibles seems to be linked to the minimal number of arrows to produce an irreducible diagram. 3×3 and 4×4 diagrams need two arrows whereas 5×5 diagrams already need three arrows.

The irreducible Krajewski diagrams are a vital ingredient for the classification of almost-commutative geometries from the view point of a particle physicist. From this classification follows the prominent position of the standard model within the Yang-Mills-Higgs theories compatible with noncommutative geometry. To underline the virtue of the algorithm we would like to present the diagram corresponding to the standard model of particle physics:



The Krajewsiki diagram of the standard model

The actual program for the algorithm was written in C^{++}.

Acknowledgements: The authors would like to thank T. Schücker and B. Iochum for their advice and support. We gratefully acknowledge our fellowships of the Friedrich-Ebert-Stiftung.

References

[1] B. Iochum, T. Schücker, C. Stephan, "On a Classification of Irreducible Almost Commutative Geometries", hep-th/0312276, J. Math. Phys. in press
J.-H. Jureit, C. Stephan, "On a Classification of Irreducible Almost Commutative Geometries, a Second Helping", hep-th/0501134, J. Math. Phys. in press
J.-H. Jureit, T. Schücker, C. Stephan, "On a Classification of Irreducible Almost Commutative Geometries III", to be published,
T. Schücker "Krajewski diagrams and spin lifts", hep-th/0501181

[2] A. Chamseddine & A. Connes, "The spectral action principle", hep-th/9606001, Comm. Math. Phys. 182 (1996) 155

[3] A. Connes, *Noncommutative Geometry*, Academic Press, London and San Diego (1994)

[4] T. Schücker, "Forces from Geometry", hep-th/0111236

[5] T. Krajewski, “Classification of finite spectral triples”, hep-th/9701081, J. Geom. Phys. 28 (1998) 1
M. Paschke & A. Sitarz, “Discrete spectral triples and their symmetries”, q-alg/9612029, J. Math. Phys. 39 (1998) 6191

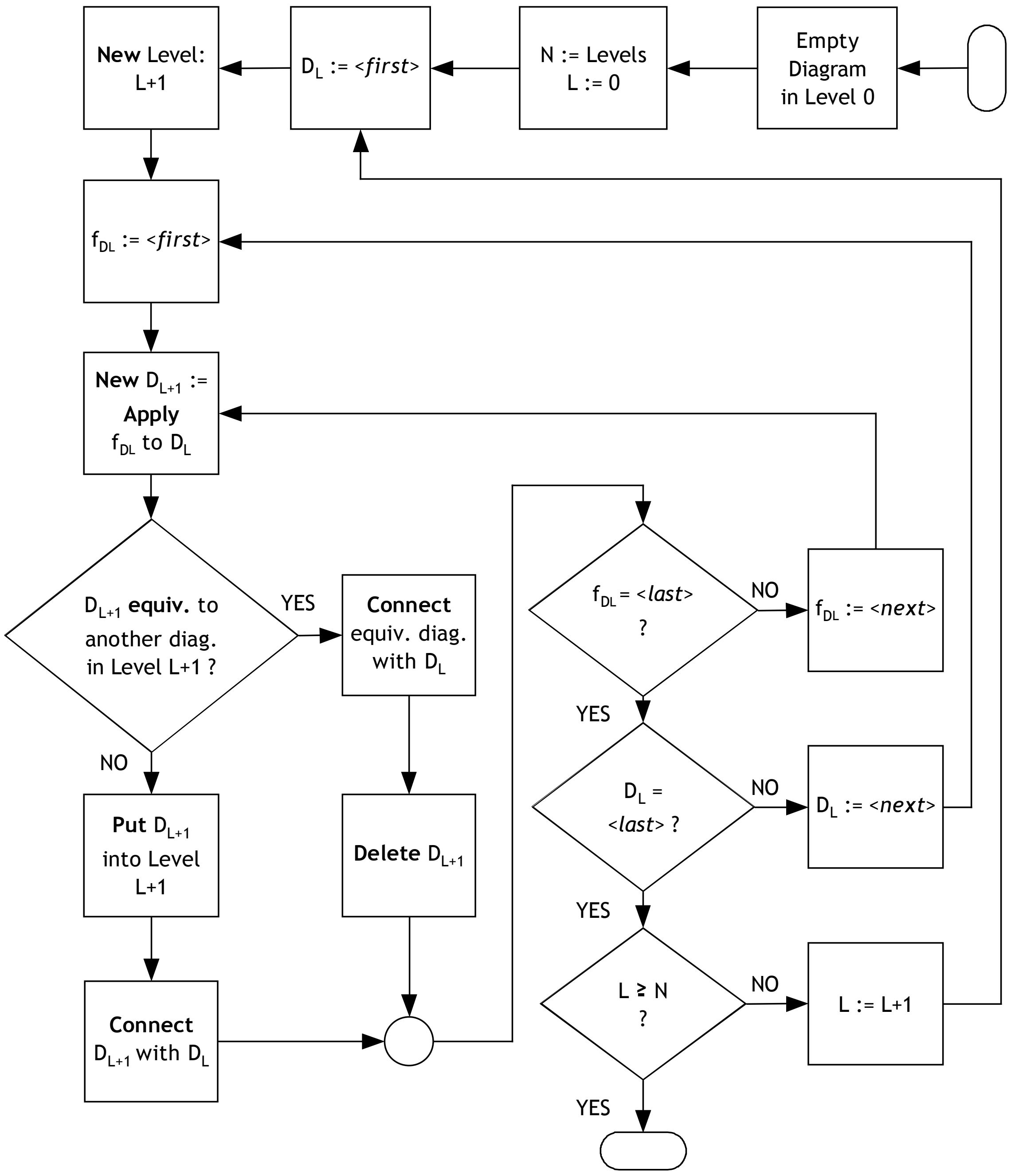


Figure 3: The Main-Net Subalgorithm

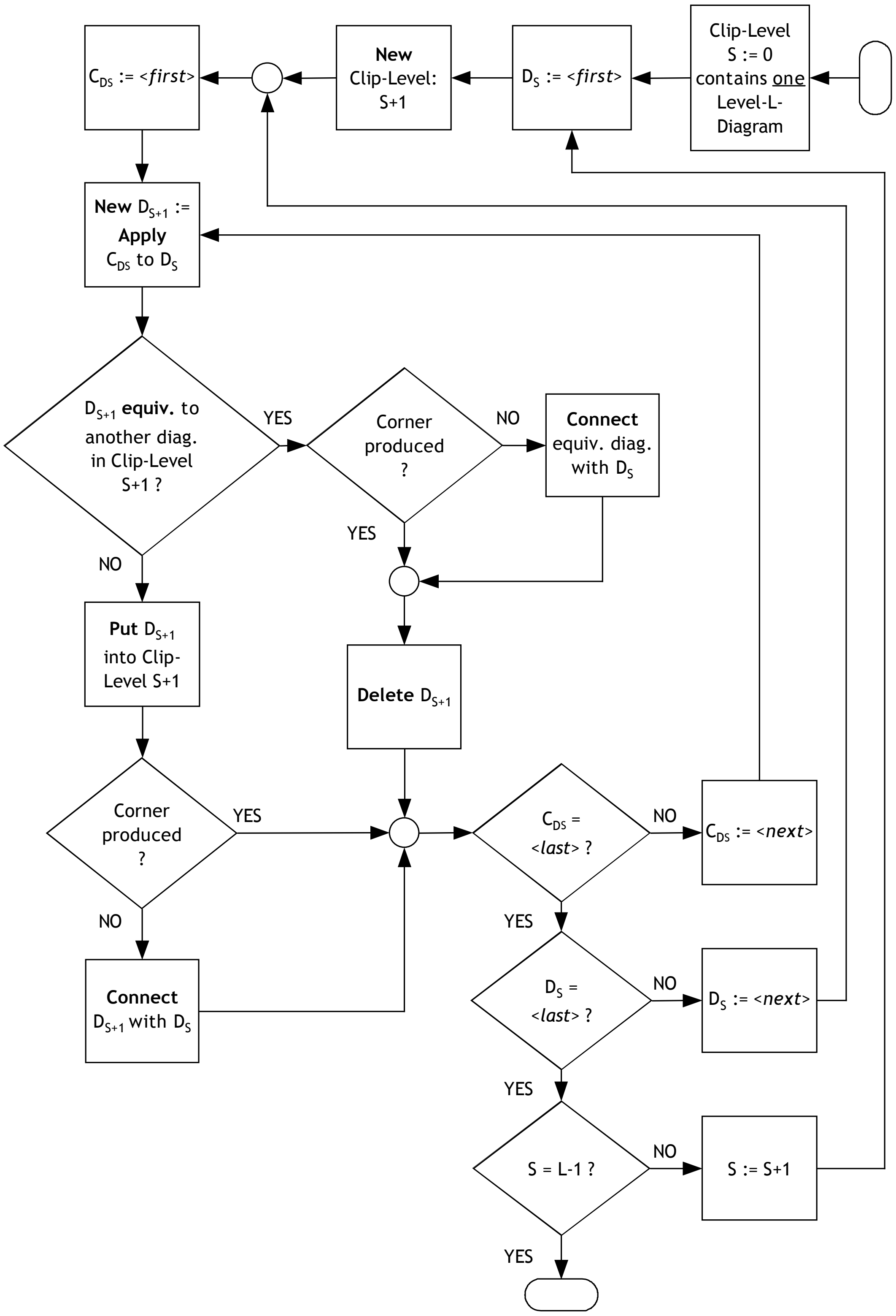

Clip-Level S := 0 contains one Level-L-Diagram
D_S := <first>
New Clip-Level: S+1
C_{DS} := <first>
New D_{S+1} := Apply C_{DS} to D_S
D_{S+1} equiv. to another diag. in Clip-Level S+1 ?
YES
Corner produced ?
NO
Connect equiv. diag. with D_S
YES
Delete D_{S+1}
NO
Put D_{S+1} into Clip-Level S+1
Corner produced ?
YES
NO
Connect D_{S+1} with D_S
C_{DS} = <last> ?
NO
C_{DS} := <next>
YES
D_S = <last> ?
NO
D_S := <next>
YES
S = L-1 ?
NO
S := S+1
YES

Figure 4: The Clip-Net Subalgorithm

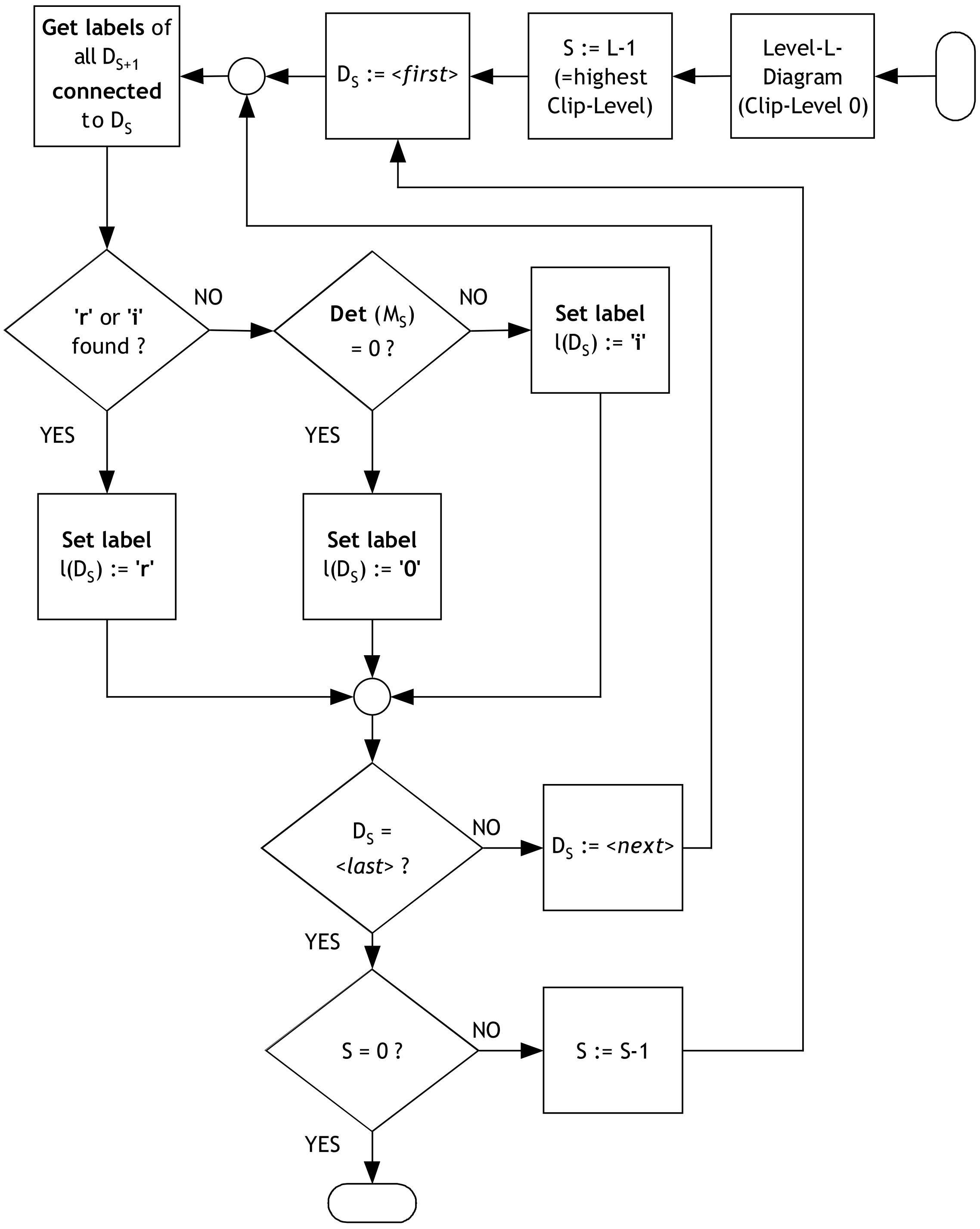

Get labels of all D_{S+1} connected to D_S
D_S := <*first*>
S := L-1 (=highest Clip-Level)
Level-L-Diagram (Clip-Level 0)
'r' or 'i' found ?
NO
Det (M_S) = 0 ?
NO
Set label l(D_S) := 'i'
YES
YES
Set label l(D_S) := 'r'
Set label l(D_S) := '0'
D_S = <*last*> ?
NO
D_S := <*next*>
YES
S = 0 ?
NO
S := S-1
YES

Figure 5: The Label Subalgorithm

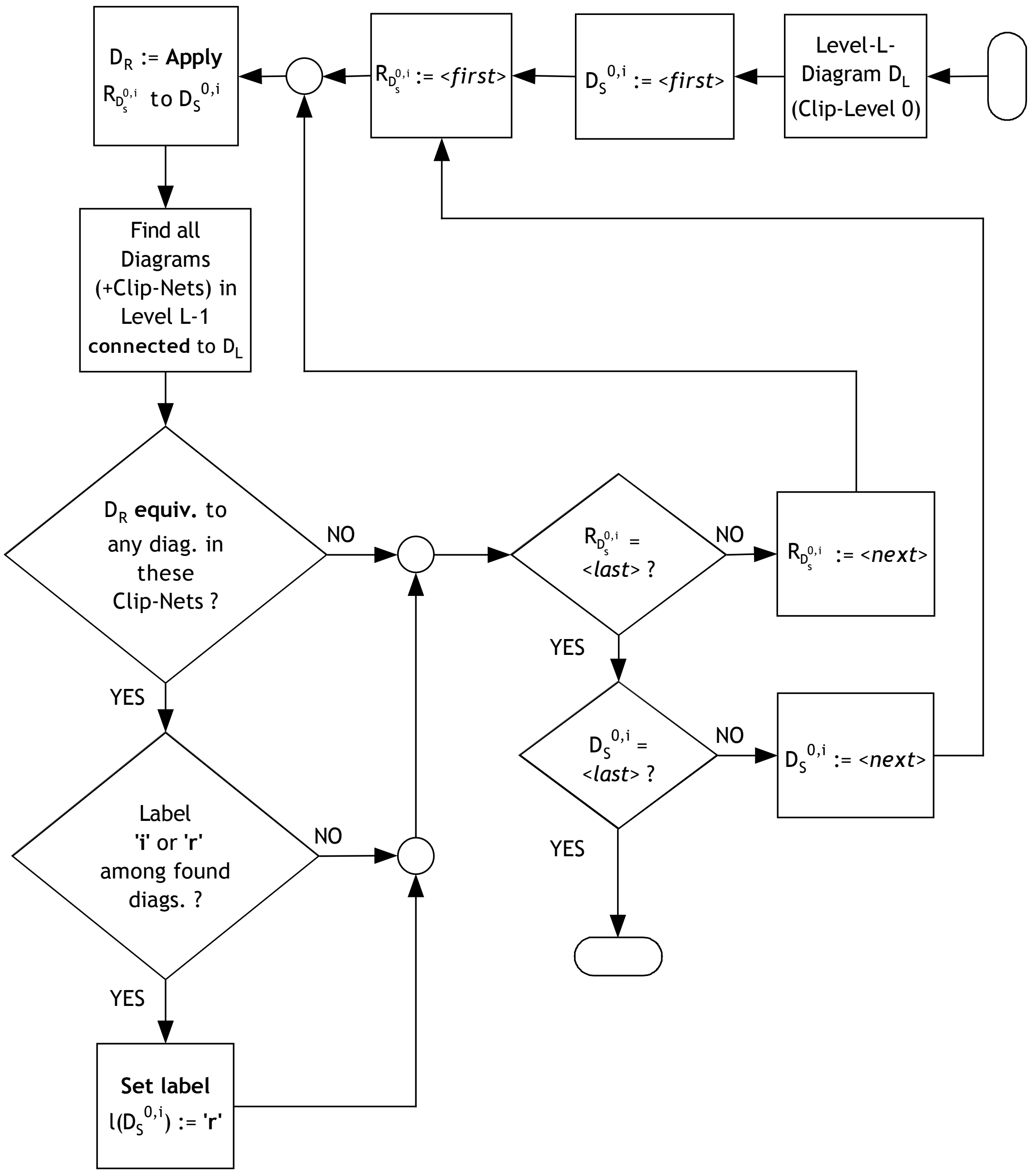

Level-L-Diagram D_L (Clip-Level 0)
$D_S^{0,i}$:= <*first*>
$R_{D_S^{0,i}}$:= <*first*>
D_R := **Apply** $R_{D_S^{0,i}}$ to $D_S^{0,i}$
Find all Diagrams (+Clip-Nets) in Level L-1 **connected** to D_L
D_R **equiv.** to any diag. in these Clip-Nets ?
NO
YES
Label **'i'** or **'r'** among found diags. ?
NO
YES
Set label $l(D_S^{0,i})$:= **'r'**
$R_{D_S^{0,i}}$ = <*last*> ?
NO
YES
$R_{D_S^{0,i}}$:= <*next*>
$D_S^{0,i}$ = <*last*> ?
NO
YES
$D_S^{0,i}$:= <*next*>

Figure 6: The Label Correction Subalgorithm